

Two Forbidden Induced Minor Theorems for Antimatroids

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Abstract

Antimatroids were discovered by Dilworth in the context of lattices [4] and introduced by Edelman and Jamison as convex geometries in [5]. The author of the current paper independently discovered (possibly infinite) antimatroids in the context of proof systems in mathematical logic [1]. Carlson, a logician, makes implicit use of this view of proof systems as possibly infinite antimatroids in [2]. Though antimatroids are in a sense dual to matroids, far fewer antimatroid forbidden minor theorems are known. Some results of this form are proved in [6], [7], [8], and [9]. This paper proves two forbidden induced minor theorems for these objects, which we think of as proof systems.

Our first main theorem gives a new proof of the forbidden induced minor characterization of partial orders as proof systems, proved in [8] in the finite case and stated in [10] for what we call strong aut descendable proof systems. It essentially states that, pathologies aside, there is a certain unique simplest nonposet. Our second main theorem states the new result that, pathologies aside, there is a certain unique simplest proof system containing points x and y such that x needs y in one context, yet y needs x in another.

1. Introduction

The following definition of *autonomous system* is basic to all that follows. An autonomous system is essentially an abstract proof system, though this is

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likely not obvious from the definition. We refer the curious reader to [1], in which this intuition is explained in detail.

Definition 1. An autonomous system is a set S together with a set T of subsets of S , called autonomous sets, satisfying the following two conditions:

- (i) T is closed under arbitrary union.
- (ii) (The Order Property) For every autonomous set A there is a total order \leq' of A such that every \leq' downward closed set is autonomous.

Note that since the union of autonomous sets is autonomous, there is a largest autonomous subset S' of S . Since all the structure is contained within S' , we assume unless otherwise noted that $S = S'$. More generally, every set X in an autonomous system has a largest autonomous subset A under inclusion. We call A the autonomous part of X . We call $X - A$ the nonautonomous part of X .

2. The Canonical Orders

The canonical orders may be thought of as context dependent orders of needing. For an autonomous set A , we think of $x <_A y$ as saying that y needs x if one is restricted to only using tools from the set A . If the notion of proof is defined abstractly, then $x <_A y$ means that x precedes y in every proof whose underlying set is a subset of A . Since the definition of proof is outside the scope of this paper, we take the following as our definition, though it is normally a proposition.

Definition 2. Let A be autonomous and let x, y be in A . Then $x \leq_A y$ iff every autonomous subset of A containing y also contains x .

We need several lemmas.

Lemma 3. *Let S be an autonomous system, let x be in S , and let A be a minimal autonomous subset of S containing x . Then x is a maximum element of the canonical order \leq_A .*

PROOF. Suppose not. Then there is y in A such that $y \not\prec_A x$. Therefore there is an autonomous subset B of A containing x and not y . So A is not a minimal autonomous set containing x , a contradiction.

Lemma 4. *Let $A \subseteq B$ be autonomous sets and $x, y \in A$. If $x <_B y$ then $x <_A y$.*

PROOF. Suppose $x <_B y$. Then every autonomous subset of B containing y also contains x . In particular, every autonomous subset of A containing y also contains x . Therefore $x <_A y$.

Lemma 5. *Let A be autonomous and B an autonomous subset of A . Then B is downward closed in the canonical order \leq_A .*

PROOF. We have to show B is \leq_A downward closed, so let $y \in B$ and $x <_A y$. We must show $x \in B$. Since $x <_A y$, we see every autonomous subset of A containing y also contains x . In particular, B contains x .

3. Partial Orders As Autonomous Systems

Lemma 6. *Let (P, \leq) be a partial order. Then the set T of downward closed sets is closed under arbitrary union and satisfies the order property.*

PROOF. To see T is closed under arbitrary union, let D_i be downward closed for each i in an index set I . We must show $D = \bigcup_{i \in I} D_i$ is downward closed. Let $y \in D$ and $x < y$. Since y is in D then y is in some D_i . Since D_i is downward closed and $x < y$, we see that x is in D_i . Since $D_i \subseteq D$, we see that x is in D .

To see that T satisfies the order property, let $D \in T$ be a downward closed set. We must show there is a total order \leq' on D such that every \leq' downward closed set is in T . In other words, every \leq' downward closed set must be \leq downward closed. This means exactly that $x \leq y$ implies $x \leq' y$ for all x, y in P . Such a total order is called a linear extension of \leq . Linear extensions are well known to exist for every partial order. The proof is thus complete.

Corollary 7. *Let (P, \leq) be a partial order and let T be the set of \leq downward closed sets. Then (P, T) is an autonomous system.*

The next lemma is instrumental in proving a useful characterization of partial orders as proof systems.

Lemma 8. *Let A be autonomous and S an arbitrary subset of A . Then S is \leq_A downward closed iff it is the (possibly infinite) intersection of autonomous subsets of A .*

PROOF. Let us first see that if S_i is an autonomous subset of A for all $i \in I$ then $\bigcap_{i \in I} S_i$ is \leq_A downward closed. Each S_i is by hypothesis an autonomous subset of A and so is \leq_A downward closed by Lemma 5. Since downward closed subsets of an arbitrary partial order are closed under intersection, in particular so are the \leq_A downward closed subsets. It follows that $\bigcap_{i \in I} S_i$ is \leq_A downward closed.

Now, for the nontrivial direction. We must show every \leq_A downward closed set S can be represented as $\bigcap_{i \in I} S_i$ for some autonomous subsets S_i of A . It is enough to show S is the intersection of all autonomous subsets of A containing it, so let I index all these sets S_i . It is obvious S is a subset of the intersection of all autonomous subsets of A containing it, so we have only to show the reverse inclusion.

So we have to show the intersection $\bigcap_{i \in I} S_i$ is contained in S , which means we must show every element of $\bigcap_{i \in I} S_i$ is also an element of S . We show the contrapositive, that given $x \in A$, if x is not an element of S then x is not an element of $\bigcap_{i \in I} S_i$.

So take x not in S . To show x is not in $\bigcap_{i \in I} S_i$ is to show there is $i \in I$ such that x is not in S_i . Since our S_i 's are all the autonomous subsets of A containing S , this means we have to give an autonomous subset of A containing S but not containing x . It is enough to give, for each $s \in S$, an autonomous subset B of A containing s and not x . For then $\bigcup_{s \in S} B_s$ will be the desired autonomous subset of A containing S and not x , completing the proof.

So take $s \in S$. How do show there is an autonomous subset of A containing s and not x ? If there were no such autonomous subset of A , then every autonomous subset of A containing s would also contain x , and therefore we would have $x \leq_A s$. Now s is in S and S is \leq_A downward closed by hypothesis, which implies x is in S , contrary to our choice of x as an element not in S . This contradiction proves the lemma.

The following theorem characterizes partial orders in terms of autonomous sets.

Theorem 9. *Let (P, T) be an autonomous system given by autonomous sets. Then the following are equivalent:*

- (i) (P, T) is a partial order.
- (ii) The T autonomous sets are closed under arbitrary intersection.

PROOF. (i) \Rightarrow (ii): If (P, T) is a partial order then we may take \leq on P such that the sets in T are exactly the \leq downward closed sets. Since the downward closed sets of a partial order are closed under arbitrary intersection, we see the autonomous sets of (P, T) are as well.

(ii) \Rightarrow (i): For the converse, we assume the T autonomous sets are closed under arbitrary intersection. By Lemma 8, the \leq_P downward closed subsets of P are exactly the intersections of autonomous subsets of P . Since we are assuming the arbitrary intersection of autonomous sets is autonomous, this implies the \leq_P downward closed sets are exactly the T autonomous sets. Therefore (P, T) is a partial order as claimed.

4. Deletion, Contraction, Quotients, and Minors

We now rigorously define the containment relations for autonomous systems with which our main theorems are stated. We first define deletions and contractions.

Definition 10. Let (P, T) be an autonomous system and C a subset of P . Then the contraction P/C of P to $P - C$ is defined as the autonomous system $(P - C, T')$ with domain $P - C$ and set of autonomous sets

$$T' = \{B \subseteq C : B = A - C \text{ for some } A \text{ in } T\}$$

Definition 11. Let (P, T) be an autonomous system and let C be a subset of P . Then the deletion $P \setminus C$ of P to $P - C$ is defined as the autonomous system $(P - C, T')$ with domain $P - C$ and set of autonomous sets

$$T' = \{B \subseteq C : B \in T\}$$

Note that we sometimes denote P/C instead by $P|(P - C)$ and refer to restricting P to $P - C$. Similarly, we sometimes denote $P \setminus C$ by $P.(P - C)$ and refer to dotting to $P - C$. We refer to an autonomous system obtained from P by a sequence of deletions and contractions (equivalently a sequence of dottings and restrictions) as a subdot or delecontraction.

We now define homomorphisms and quotients. These notions, together with deletions and contractions, will allow us to define minors and induced minors.

Definition 12. An autonomous system homomorphism is a function f from an autonomous system P to an autonomous system Q such that $f^{-1}(A)$ is autonomous in P for all autonomous $A \subseteq Q$.

Definition 13. Let (P, T_P) and (Q, T_Q) be autonomous systems. A surjective autonomous system homomorphism $f : (P, T_P) \rightarrow (Q, T_Q)$ is called a quotient map if $T' \subseteq T_Q$ for all autonomous systems (Q, T') with domain Q such that $f : (P, T_P) \rightarrow (Q, T')$ is a homomorphism. The autonomous system (Q, T_Q) is then called a quotient of (P, T_P) .

Definition 14. Let Q and Q' be autonomous systems. We say that Q' is an induced minor of Q if there is a sequence $Q = Q_1, \dots, Q_n = Q'$ of autonomous systems such that for each i with $1 \leq i < n$, one of the following conditions holds:

- (i) $Q_{i+1} = Q_i/C$ for some subset C of Q_i .
- (ii) $Q_{i+1} = Q_i \setminus C$ for some subset C of Q_i .
- (iii) Q_{i+1} is a quotient of Q_i .

If we replace the third condition with the condition that Q_{i+1} is simply a homomorphic image of Q_i , we get the notion of autonomous system minor.

The names minor and induced minor are chosen for good reason. Though technical and outside the scope of this paper, roughly speaking, it can be shown that considering each graph as a family of autonomous systems, graph minor and induced minor correspond to autonomous system minor and induced minor, respectively. Allowing homomorphic images that are not quotient maps corresponds to graph edge deletion.

5. Autonomous System Join

Joins of autonomous systems allow us to prove that when an equivalence relation is homomorphism induced, it is in fact induced by a quotient map. We recall that given a partial order (Z, \leq) and points x, y in Z , the join $x \vee y$ of x and y is the least upper bound of x and y if one exists. Otherwise $x \vee y$ is undefined. More generally, if S is a nonempty subset of Z , then $\bigvee_{x \in S} x$ is a least upper bound of S if one exists and is otherwise undefined.

Given a set $\{(P_i, T_i)\}_{i \in I}$ of autonomous systems, we let $P = \bigcup_{i \in I} P_i$ and consider the autonomous systems (P, T_i) for i in I . We let $(P, T_i) \leq (P, T_j)$ if $T_i \subseteq T_j$. With this definition of \leq , we may then speak of the least upper bound, or join, of a nonempty set of autonomous systems. The next lemma shows that the join of every nonempty set of autonomous systems exists.

Lemma 15. *Given a set of autonomous systems (P_i, T_i) for i in a nonempty index set I , the join $\bigvee_{i \in I} (P_i, T_i)$ exists. Specifically, it is the autonomous system (P, T) on $P = \bigcup_{i \in I} P_i$, where T is the closure of $\bigcup_{i \in I} T_i$ under arbitrary union.*

PROOF. First, we consider each autonomous system (P_i, T_i) instead as the autonomous system (P, T_i) . As in the statement of the lemma, we let T be the set of subsets of P of the form $\bigcup_{i \in I} A_i$, where each A_i is a (possibly empty) set in T_i . From the definition of T and the fact that autonomous sets in an autonomous system are closed under arbitrary union, it is immediate that if (P, T) is an autonomous system, then it is in fact the least upper bound as required. To show that (P, T) is an autonomous system, we must show T is closed under arbitrary union and satisfies the order property.

Closure under arbitrary union is immediate by definition of T . To show the order property, we must show every set A in T can be totally ordered such that each downward closed set is also in T . We know that $A = \bigcup_{i \in I} A_i$ for sets A_i in T_i . Since (P, T_i) is an autonomous system for each i in I , we see that each T_i satisfies the order property. We may therefore totally order each A_i as \leq_i such that each \leq_i downward closed set is autonomous.

Choose a well order \leq_w on the set I . We define a total order \leq on A as follows. For each x in A , let r_x be the \leq_w least element of I such that x is in A_{r_x} . If $r_x <_w r_y$ then let $x < y$. If $r_x = r_y = i$ then let $x \leq y$ iff $x \leq_i y$. The reader may check that this is a well defined total order on A such that each \leq downward closed set D is the union of \leq_i downward closed sets D_i for each i . Since each D_i is \leq_i downward closed, it follows that D_i is T_i autonomous. Since $T_i \subseteq T$, we see that D_i is T autonomous. Since T is closed under arbitrary union, it follows that $D = \bigcup_{i \in I} D_i$ is T autonomous. This completes the proof.

Definition 16. Let $f : P \rightarrow Q$ be an autonomous system homomorphism. We say the equivalence relation \sim on P such that $x \sim y$ iff $f(x) = f(y)$ is induced by f . We call an equivalence relation on P homomorphism induced if it is induced by some surjective homomorphism $f : P \rightarrow Q$ such that Q has at least one nonempty autonomous set.

The requirement that Q has at least one nonempty autonomous set is given because otherwise every equivalence relation would vacuously be homomorphism induced by a map to an autonomous system with only the empty set autonomous.

Lemma 17. *If $f : P \rightarrow (Q, T_i)$ is an autonomous system homomorphism for each i in a nonempty index set I , then $f : P \rightarrow \bigvee_{i \in I} (Q, T_i)$ is a homomorphism.*

PROOF. We must show the inverse image of every $\bigvee_{i \in I} (Q, T_i)$ autonomous set is P autonomous, so choose such a set A . Then $A = \bigcup_{i \in I} A_i$ for some sets A_i in T_i . For each i , $f : P \rightarrow (Q, T_i)$ is a homomorphism and therefore the set $f^{-1}(A_i)$ is P autonomous. Since the P autonomous sets are closed under

arbitrary union, we see that

$$f^{-1}(A) = f^{-1}\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} f^{-1}(A_i)$$

is P autonomous as needed.

Lemma 18. *If P is an autonomous system, Q is a set, f a function from P to Q , and $\{(Q, T_i) : i \in I\}$ is the set of all autonomous systems on Q such that $f : P \rightarrow (Q, T_i)$ is an autonomous system homomorphism, then $f : P \rightarrow \bigvee_{i \in I} (Q, T_i)$ is a quotient map.*

PROOF. Let $(Q, T) = \bigvee_{i \in I} (Q, T_i)$. It is only to show that if $f : P \rightarrow (Q, T')$ is an autonomous system homomorphism then $T' \subset T$. This is immediate from the definition of join and (Q, T) .

Corollary 19. *If an equivalence relation on an autonomous system is homomorphism induced, then it is induced by a quotient map.*

If $f : P \rightarrow (Q, T_1)$ and $f : P \rightarrow (Q, T_2)$ are quotient maps, then $T_1 \subseteq T_2 \subseteq T_1$ by definition of quotient map, so $T_1 = T_2$. Thus the quotient map of the previous corollary is unique. We may thus refer to the quotient P/\sim for any homomorphism induced equivalence relation on P . The reader should note that P/\sim for an equivalence relation \sim and P/X for a subset X of P are distinct notions.

6. Strong Aut Descendability

Our main theorems will be stated for the class of strong aut descendable autonomous systems, which includes both finite autonomous systems and arbitrary partial orders. The reader who is content to consider finite systems may skip this section and insert “finite” everywhere he or she reads strong aut descendable.

Definition 20. An autonomous system is strong aut descendable if the intersection of every chain of autonomous sets under inclusion is also autonomous.

The condition is meant to allow infinite autonomous systems while excluding certain pathologies that prevent many statements from being true for arbitrary autonomous systems. The simplest example is an autonomous system on a set $S \cup x$ such that S is an infinite set not containing x , every subset of S is autonomous, and every cofinite subset of $S \cup x$ is autonomous. The reader may show that this is an autonomous system. It is somewhat of an all purpose counterexample in the sense that for each statement we make only assuming strong aut descendability, the counterexample to the general statement is very similar in spirit to that just described.

The proof of the following lemma may be taken as a simple exercise in Zorn's Lemma.

Lemma 21. *If A is an autonomous set in a strong aut descendable autonomous system and x is a point in A , then there is a minimal autonomous subset B of A containing x .*

We need to show that strong aut descendability is preserved under taking subdots.

Lemma 22. *Strong aut descendability is preserved under taking subdots.*

PROOF. We must show that strong aut descendability is preserved under dotting and restricting. Let P be an autonomous system. Dotting yields an autonomous system of the form PA , where $A \subseteq P$ is autonomous. Since P is strong aut descendable, the intersection of every chain of subsets of P is autonomous in P . In particular, the intersection of every chain of subsets of A is a subset of A that is autonomous in P . The intersection is therefore autonomous in PA , proving that PA is strong aut descendable.

To show that strong aut descendability is preserved under restriction, let X be an arbitrary subset of P and consider a chain (I, \leq) of $P|X$ autonomous sets $\{X_i\}_{i \in I}$ under inclusion such that $i < j$ iff $X_i \subset X_j$. By definition of restriction, each X_i has the form $A_i \cap X$ for some P autonomous set A_i . Given i , let

$$B_i = \bigcup_{j \leq i} A_j.$$

Then each B_i is P autonomous as well since the autonomous sets are closed under arbitrary union. Moreover, $\{B_i\}_{i \in I}$ is a chain under inclusion so that $\bigcap_{i \in I} B_i$ is autonomous by strong aut descendability of P . Therefore

$$X \cap \left(\bigcap_{i \in I} B_i \right) = X \cap \left(\bigcap_{i \in I} \bigcup_{j \leq i} A_j \right) = X \cap \left(\bigcap_{i \in I} A_i \right) = \bigcap_{i \in I} (X \cap A_i) = \bigcap_{i \in I} X_i$$

is $P|X$ autonomous as needed.

7. The Main Theorems

The author showed in his doctoral thesis that partial orders comprise an induced minor closed class of autonomous systems. We omit the somewhat technical proof.

Lemma 23. *If (P, \leq) is a partial order and (Q, T_Q) is an induced minor of P , then Q is a partial order.*

Knowing that partial orders form an induced minor closed class, it is natural to seek a forbidden induced minor characterization of this class. The main step in proving such a theorem is Lemma 24. It is stated in terms of the autonomous system P_3 . In general, for $n \geq 1$, the autonomous system P_n has the vertex set $\{v_1, \dots, v_n\}$ of an n point path as its underlying set, with a subset of P_n autonomous iff it has the form $\{v_1, \dots, v_i\} \cup \{v_k, \dots, v_n\}$ with $0 \leq i \leq n$ and $1 \leq k \leq n+1$. In other words, the autonomous sets are the unions of paths starting at the endpoints and empty paths.

Lemma 24. *Let P be a strong aut descendable autonomous system with autonomous sets A and B such that $A \cap B$ is not autonomous. Then P contains P_3 as a subdot.*

PROOF. By dotting to $A \cup B$ if necessary, we may assume $P = A \cup B$. Since $A \cap B$ is not autonomous, we may choose x in the nonautonomous part of $A \cap B$. Let $A' = (A - B) \cup \{x\}$, $B' = (B - A) \cup \{x\}$, let $S = (A' \cup B' \cup \{x\})$, and let

$P' = P|S$. Since $A' = A \cap S$ and $B' = B \cap S$, we see by definition of P' that A' and B' are autonomous in P' . If $\{x\} = A' \cap B'$ is autonomous in P' then by definition of P' as a restriction, P must contain an autonomous subset of $A \cap B$ containing x , contrary to choice of x in the nonautonomous part. This contradiction shows $\{x\} = A' \cap B'$ is not autonomous in P' .

Since P' is strong aut descendable by Lemma 22, we may choose a minimal P' autonomous subset A'' of A' containing x . Similarly, we may choose a minimal P' autonomous subset B'' of B' containing x . Note that $A'' \cup B''$ is P' autonomous. Let $P'' = P|(A'' \cup B'')$. Then A'' and B'' are P'' autonomous sets such that $A'' \cap B'' = \{x\}$ is not P'' autonomous.

Note that since A'' is a minimal P' autonomous subset of A' containing x , it follows that x is a $\leq_{A''}$ maximum element of A'' in the autonomous system P' . Since P' and P'' have the same autonomous subsets of A'' , it follows that x is a $\leq_{A''}$ maximum element of A'' in the autonomous system P'' as well. Since $\{x\}$ is not autonomous in P'' , it follows that there is a in A'' such that $a <_{A''} x$ in the autonomous system P'' . Similarly, there is b in B'' such that $b <_{B''} x$ in the autonomous system P'' .

Let $P''' = P''|\{a, b, x\}$. Simple checking of the autonomous sets of P''' shows that P''' is isomorphic to P_3 . Obviously, P''' is a subdot of P .

Lemma 25. *If a strong aut descendable autonomous system P is not a partial order, then there are two autonomous sets whose intersection is not autonomous.*

PROOF. Since P is not a partial order, it follows from Theorem 9 that the autonomous sets are not closed under arbitrary intersection. Let $\{A_i\}_{i < \lambda}$ be a family of autonomous sets in P whose intersection is not autonomous, for some finite or infinite cardinal λ . For each $i < \lambda$, let

$$B_i = \bigcap_{j \leq i} A_j.$$

Note that the B_i 's comprise a descending chain of sets under inclusion whose intersection is not autonomous. Therefore some B_i is not autonomous. Choose

the least such i . Then i can not be a limit ordinal, for then

$$B_i = \bigcap_{j \leq i} A_j = \bigcap_{j < i} A_j,$$

so B_i would be the intersection of a chain of autonomous sets and therefore autonomous.

So the least such i must be a successor ordinal. That is, $i = j + 1$. Therefore $B_i = B_j \cap A_{j+1}$. Since B_j and A_{j+1} are autonomous but B_i is not, the proof is complete.

Corollary 26. *A strong aut descendable autonomous system is a partial order iff it has no P_3 subdot.*

PROOF. We know the subdot of a partial order is a partial order, which is therefore not P_3 as P_3 contains the autonomous sets $\{v_1, v_2\}$ and $\{v_3, v_2\}$ whose intersection is not autonomous. Inversely, if a strong aut descendable autonomous system P is not a partial order, then by Lemma 25 there are autonomous sets A and B such that $A \cap B$ is not autonomous. By Lemma 24, it follows the autonomous system has a P_3 subdot.

Corollary 27. *A strong aut descendable autonomous system is a partial order iff it has no P_3 induced minor.*

PROOF. Immediate from the fact that a subdot is a minor and the fact that the pure contraction of a partial order is a partial order.

Consider the autonomous system P_4 with vertices a, x, y, b in that order in the path. Note that $x <_{\{a, x, y\}} y$ and $y <_{\{b, y, x\}} x$. In fact, P_4 is the simplest autonomous system exhibiting such behavior in the sense that every strong aut descendable autonomous system containing points x and y and sets A and B such that $x <_A y$ and $y <_B x$ contains a P_4 induced minor. We prove this claim now.

The proof that follows is very similar in spirit to the proof of Lemma 24, which suggests that the following theorem and Corollary 26 may be combined.

In fact Corollary 26 and the following theorem may be seen as corollaries of the same result, but that result is far more technical to state and prove, so we treat each separately.

Theorem 28. *Let P be a strong aut descendable autonomous system with autonomous sets A and B and points x and y such that $x <_A y$ and $y <_B x$. Then P contains a P_4 induced minor.*

PROOF. By dotting to $A \cup B$ if necessary, we may assume $P = A \cup B$. Given such A , B , x , and y , let $A' = (A - B) \cup \{x, y\}$, $B' = (B - A) \cup \{x, y\}$, let $S = (A' \cup B' \cup \{x, y\})$, and let $P' = P|S$. Since $A' = A \cap S$ and $B' = B \cap S$, we see by definition of P' that A' and B' are both autonomous sets in P' containing x and y . If there is a P' autonomous subset C of A' containing y and not x then $C = D \cap S$ for some P autonomous set D . But then $D \subseteq A$, contrary to the fact that there is no P autonomous subset of A containing y and not x . This contradiction shows that every P' autonomous subset of A' containing y also contains x . Therefore $x <_{A'} y$ in the autonomous system P' . Similarly, $y <_{B'} x$ in the autonomous system P' .

Since P' is strong aut descendable by Lemma 22, we may choose a minimal P' autonomous subset A'' of A' containing y . Similarly, we may choose a minimal P' autonomous subset B'' of B' containing x . $A'' \cup B''$ is P' autonomous. Let $P'' = P|(A'' \cup B'')$. Then A'' and B'' are P'' autonomous sets such that $A'' \cap B'' = \{x, y\}$.

Note that since A'' is a minimal P' autonomous subset of A' containing y , it follows that y is a $\leq_{A''}$ maximum element of A'' in the autonomous system P' . In particular, $x <_{A''} y$ in the autonomous system P' . Since P' and P'' contain the same autonomous subsets of A'' , we see that $x <_{A''} y$ in the autonomous system P'' as well. Similarly, $y <_{B''} x$ in the autonomous system P'' . Since $x <_{A''} y$ in P'' , we see that y is not a P'' axiom. Therefore $B'' - \{x, y\}$ is nonempty. Similarly, $A'' - \{x, y\}$ is nonempty.

The reader may check that the partition of P'' with cells $A'' - \{x, y\}$, $B'' - \{x, y\}$, $\{x\}$, and $\{y\}$ is homomorphism induced. Let P''' be the pure contraction

of P'' with respect to this partition. The reader may check that this autonomous system is isomorphic to P_4 .

We note that while our forbidden induced minor theorem for partial orders can also be seen as a forbidden subdot theorem, the same is not true for the theorem just stated. Let P be the six point autonomous system on $\{a_1, a_2, x, y, b_1, b_2\}$ whose autonomous sets are the (possibly empty) unions of the sets $\{a_1\}, \{a_2\}, \{b_1\}, \{b_2\}, \{a_1, x\}, \{a_2, x\}, \{b_1, y\}, \{b_2, y\}, A := \{a_1, a_2, x, y\}$, and $B := \{b_1, b_2, y, x\}$. The reader may show that this is an autonomous system for which $x <_A y$ and $y <_B x$, yet P has no P_4 subdot.

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